MATH4060 Tutorial 1

19 January 2023

Problem 1. *Show that*

(i) $f(z) = e^{-\pi z^2}$ *belongs to* $\mathfrak{F}_a$ *for all* $a > 0$.

(ii) $f(z) = \frac{c}{c^2+z^2}$ *belongs to* $\mathfrak{F}_a$ *for all* $0 < a < c$.

For (i), f is holomorphic on $S_a = \{|\operatorname{Im}(z)| < a\}$ because it is entire. If $z = x + iy \in S_a$,

$$|e^{-\pi z^2}| = e^{\operatorname{Re}(-\pi z^2)} = e^{-\pi(x^2-y^2)} \leq Ce^{-\pi x^2}$$

for some constant $C > 0$ independent of $|y| < a$ (e.g. take $C = e^{\pi a^2}$). So it suffices to show that there exists $A > 0$ such that $e^{-\pi x^2} \leq A/(1 + x^2)$ for all $x \in \mathbb{R}$, i.e. that $(1 + x^2)e^{-\pi x^2}$ is bounded. Note that a continuous function $g(x)$ on $\mathbb{R}$ is bounded if $\lim_{x\to\pm\infty} g(x)$ exist. And indeed, by applying L'hopital's rule twice, we have

$$\lim_{x\to\infty} \frac{1 + x^2}{e^{\pi x^2}} = 0.$$

For (ii), since $0 < a < c$, the function $f(z) = c/(c^2 + a^2)$ is holomorphic on S_a (as it does not contain the poles $\pm ci$). If $z = x + iy$,

$$|c^2 + z^2| = |(c^2 + x^2 - y^2) + 2xyi| \geq |c^2 + x^2 - y^2| > (c^2 - a^2) + x^2.$$

So

$$|f(z)| \leq \frac{c}{(c^2 - a^2) + x^2}.$$

As above, this is bounded by some $A/(1+x^2)$ because the RHS is a continuous function in x and

$$\lim_{x\to\pm\infty} \frac{c(1 + x^2)}{(c^2 - a^2) + x^2} = c < \infty.$$

Problem 2. *Let* f *be non-constant and holomorphic in an open set containing the closed unit disc. Show that if* $\left|f(z)\right| = 1$ *whenever* $|z| = 1$*, then the image of* f *contains the unit disc.*

We first show that f has a zero in $\mathbb{D}$. Suppose for a contradiction, f is nowhere vanishing on $\mathbb{D}$. Applying the maximum-modulus theorem to f and $1/f$ (both holomorphic in $\mathbb{D}$), shows that $|f(z)| \leq 1$ and $1/|f(z)| \leq 1$ on $\mathbb{D}$, so $|f(z)| = 1$ and f must be constant, a contradiction. Next, we show that $f(z) - w = 0$ also has a solution for any $w \in \mathbb{D}$. We can use Rouche's theorem on the unit circle (with w considered as a constant function) because $\left|f(z)\right| = 1$ on $|z| = 1$ and $|w| < 1$. This shows that $f(z)$ and $f(z) - w$ have the same number of zeros in $\mathbb{D}$. Thus the image of f contains any such w.

Problem 3. *Find the relation between two entire functions* $f(z)$ *and* $g(z)$ *satisfying* $\left|f(z)\right| \leq C(1 + |z|^k)\left|g(z)\right|$ *for some constant* $C > 0$ *and non-negative integer* k.

Assume $g \not\equiv 0$. We want to show that f is a polynomial multiple of g. First define $h(z) = f(z)/g(z)$, which is holomorphic on $\mathbb{C} \setminus \{\text{zeros of } g\}$ and satisfies $\left|h(z)\right| \leq C(1 + |z|^k)$.

Observe that $h(z)$ extends to an entire function on $\mathbb{C}$. Indeed, if g is a non-zero constant function, then h is trivially entire; otherwise, g has a discrete zero set S, forming the (isolated) singularities of h. For each $z_0 \in S$, $h(z)$ is holomorphic and bounded on $0 < |z - z_0| \leq r$ for some small $r > 0$. By Riemann theorem, each z_0 is a removable singularity. Hence $h(z)$ is entire. Next we claim that any entire function satisfying $\left|h(z)\right| \leq C(1 + |z|^k)$ on $\mathbb{C}$ is a polynomial. Write $h(z) = \sum_{n=0}^{\infty} a_n z^n$ as a power series, where $a_n = h^{(n)}(0)/n!$. As a corollary of Cauchy's integral formula, for any $R > 0$, $|a_n| \leq \sup_{|z|=R} h(z)/R^n \leq C(1 + R^k)/R^n$. If $n > k$, the RHS converges to zero as $R \to \infty$, and so $a_n = 0$ for $n > k$. In other words, h is a polynomial of degree $\leq k$, and f is a polynomial multiple of g. Note that, in particular, by taking $k = 0$, this shows that an entire function cannot be dominated by another unless they are constant multiples of each other.

Problem 4. *Prove that any injective entire function f is linear, i.e. $f(z) = az + b$ for $a \neq 0$.*

We study the type of singularity f can have at infinity. Consider $g(z) = f(1/z)$ on $\mathbb{C} \setminus \{0\}$. First, we rule out $z = 0$ as a removable singularity of g: otherwise, $g(z)$ is bounded near $z = 0$, $f(z)$ is bounded near infinity, and Liouville's theorem implies that f is constant, a contradiction to its injectivity. Next we rule out essential singularity as a possibility: otherwise take $r > 0$, then Casorati-Weierstrass theorem states that the image $g(D_r(0) \setminus \{0\})$ is dense in $\mathbb{C}$. For any open subset U of $\mathbb{C} \setminus \overline{D}_r(0)$, $g(U)$ is open by the open mapping theorem, so must contain some element of $g(D_r(0) \setminus \{0\})$. This contradicts injectivity because the two subsets of the domain are disjoint.

So $z = 0$ must be a pole. The power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ gives the Laurent expansion $g(z) = \sum_{n=0}^{\infty} a_n z^{-n}$. Because $z = 0$ is a pole, $a_n = 0$ for all large n, and so f is a polynomial. As it is injective, f cannot have more than one zero, so $f(z) = a(z - z_0)^m$. We want to show that $m = 1$. Certainly, $m \neq 0$. And if $m \geq 2$, taking $z_1 = z_0 + 1$ and $z_2 = z_0 + e^{2\pi i/m}$ gives $f(z_1) = a = f(z_2)$, so f is not injective. Therefore $m = 1$ and $a \neq 0$.